CALCULATING THE HEATING OF A MASSIVE CYLINDER BY THE SWEEP METHOD

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Inzhenerno-Fizicheskii Zhurnal, Vol. 12, No. 6, pp. 736–741, 1967

UDC 536.21:621.783

Numerical solution of the heat conduction problem for a massive cylinder with allowance for heat transfer by radiation at the cylinder surface and for the furnace time lag is considered.

Two problems of optimal control of heating of large pieces of metal were posed in [1]. The development of numerical methods of solution of these problems made it necessary to solve the heat conduction equation with allowance for radiative heat transfer, heater time lags, and the temperature dependences of the thermophysical parameters.

Methods involving explicit net schemes for the approximation of the heat conduction equation [2–4] could not be used because of the prohibitively large computer times involved. An acceptable solving algorithm was then constructed on the basis of the well-known sweep method [3].

Let us consider the problem of heating of a long cylinder with radiative and convective heat transfer at its surface. The heat conduction equation in cylindrical coordinates is

$$c(T)\gamma\frac{\partial T}{\partial\tau}=\lambda(T)\left(\frac{\partial^2 T}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial T}{\partial\rho}\right)+\frac{\partial\lambda(T)}{\partial\rho}\,\frac{\partial T}{\partial\rho},$$

$$0\leqslant\rho\leqslant R;\quad 0\leqslant\tau\leqslant\tau_0; \qquad (1)$$

the initial condition is

$$T(\rho,\,0)=T_0(\rho), \qquad (2)$$

and the boundary condition at the surface

$$\lambda(T)\frac{\partial T}{\partial\rho}\Big|_{\rho=R}=\alpha_1\left(T_{\rm f}^4-T_{\rm sur}^4\right)+\alpha_2\left(T_{\rm f}-T_{\rm sur}\right). \qquad (3)$$

Let the furnace temperature T_f be related to the function of time $u(\tau)$ [which describes either the position of the gate through which fuel is fed into the operating space of the furnace or the applied electrical power in the case of an electric surface] by the differential equation

$$\frac{dT_{\rm f}}{d\tau}=-\alpha_3\left(T_{\rm f}^4-T_{\rm sur}^4\right)-\alpha_4\left(T_{\rm f}-T_{\rm sur}\right)+$$
$$+\alpha_5\left(\Phi-T_{\rm f}\right)+\alpha_6 u(\tau) \qquad (4)$$

under the initial condition

$$T_{\rm f}(0)=T_{\rm f}^0. \qquad (5)$$

Equation (4) was obtained by considering the elementary thermal balances in the system, i.e., those between the heated body, the furnace, and the medium.

System (1)–(5) describes, for example, the heating of a cylindrical specimen in an electric muffle furnace or the heating of a cylindrical ingot in a flame furnace of the chamber type.

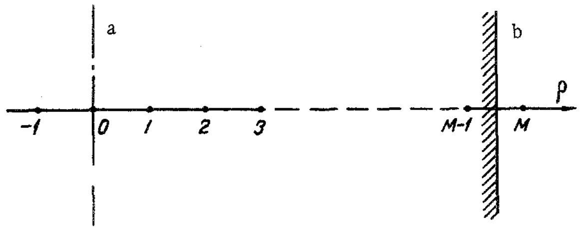

Fig. 1. Nodes of the net domain along the radius of the heated cylinder (a = axis; b = surface).

To solve the problem numerically we use an implicit net scheme [4] for system (1)–(5). The process time is broken up into (generally unequal) intervals $\Delta\tau$. As the computational net it is convenient to choose a set of points with the coordinates $\rho_m = m\Delta\rho$, where $\Delta\rho = R/(M-1.5)$; M is an integer, and $m = 0, 1, 2, \ldots, M$.

The point with the number M lies outside the cylinder and is fictitious (Fig. 1). The following relations must be fulfilled at each instant for the points $m = 1, 2, \ldots, M-1$:

$$\gamma c(T_m^k)\frac{T_m^{k+1}-T_m^k}{\Delta\tau}=$$
$$=\lambda(T_m^k)\left[\frac{T_{m-1}^{k+1}-2T_m^{k+1}+T_{m+1}^{k+1}}{\Delta\rho^2}+\frac{1}{m\Delta\rho}\times\right.$$
$$\left.\times\frac{T_{m+1}^{k+1}-T_{m-1}^{k+1}}{2\Delta\rho}\right]+\frac{\lambda(T_{m+1}^k)-\lambda(T_{m-1}^k)}{2\Delta\rho}\times$$
$$\times\frac{T_{m+1}^{k+1}-T_{m-1}^{k+1}}{2\Delta\rho},\quad m=1,\,2,\,\ldots,\,M-1. \qquad (6)$$

In order to obtain the net equation for the zero node of the net (the cylinder axis) we take the limit as $\rho\to 0$ in (4) and make use of the symmetry of the temperature distribution in the cylinder, i.e., of the fact that $T_1 = T_{-1}$. This yields the following relationship between T_0 and T_1 [4]:

$$\gamma c(T_0^k)\frac{T_0^k-T_0^{k+1}}{\Delta\tau}=2\lambda(T_0^k)\frac{2(T_1^{k+1}-T_0^{k+1})}{\Delta\rho^2}. \qquad (7)$$

System (6), (7) contains M equations and M + 1 unknowns. We obtain the other required equations from boundary condition (3) and furnace equation (4) which are sufficiently well approximated by the following relations:

$$\lambda\left(\frac{T_M^k+T_{M-1}^k}{2}\right)\frac{T_M^{k+1}-T_{M-1}^{k+1}}{\Delta\rho}=$$

$$=\alpha_1\left[(T_f^{k+1})^4-\left(\frac{T_M^{k+1}+T_{M-1}^{k+1}}{2}\right)^4\right]+$$

$$+\alpha_2\left[T_f^{k+1}-\frac{T_M^{k+1}+T_{M-1}^{k+1}}{2}\right]; \quad (8)$$

$$\frac{T_f^{k+1}-T_f^k}{\Delta\tau}=-\alpha_3\left[(T_f^{k+1})^4-\left(\frac{T_M^{k+1}+T_{M-1}^{k+1}}{2}\right)^4\right]-$$

$$-\alpha_4\left[T_f^{k+1}-\frac{T_M^{k+1}+T_{M-1}^{k+1}}{2}\right]+\alpha_5(\Phi-T_f^{k+1})+\alpha_6u^{k+1}, \quad (9)$$

where we assume that $T_{sur}^{k+1}=(T_M^{k+1}+T_{M-1}^{k+1})/2$. System of (M + 2) equations (6)–(9) contains (M + 2) unknowns T_0^{k+1}, T_1^{k+1}, ..., T_M^{k+1}, T_f^{k+1}. We shall omit the superscript (k + 1) from now on, so that $T_m^{k+1}\equiv T_m$. Following the scheme of the sweep method described in [3], we transform Eqs. (6) and (7) into

$$T_m=x_{m+1}T_{m+1}+y_{m+1};\quad m=0,\ 1,\ \ldots,\ M-1. \quad (10)$$

Here

$$x_1=1\Bigg/\left[1+\frac{\gamma c(T_0^k)\Delta\rho^2}{4\Delta\tau\lambda(T_0^k)}\right];$$

$$y_1=x_1\frac{\Delta\rho^2\gamma c(T_0^k)}{4\Delta\tau\lambda(T_0^k)}T_0^k,$$

and the coefficients x_{m+1}, y_{m+1} (m = 1, 2, ..., M − 1) are given by the following recursion formulas:

$$x_{m+1}=\frac{a_m}{2b_m-c_mx_m}; \quad (11)$$

$$y_{m+1}=\frac{c_my_m+d_m}{a_m}x_{m+1}, \quad (12)$$

where

$$a_m=\lambda(T_m^k)-\frac{1}{2m}-\frac{\lambda(T_{m+1}^k)-\lambda(T_{m-1}^k)}{4}; \quad (13)$$

$$b_m=\lambda(T_m^k)+\frac{\gamma c(T_m^k)\Delta\rho^2}{2\Delta\tau}; \quad (14)$$

$$c_m=\lambda(T_m^k)-\frac{1}{2m}-\frac{\lambda(T_{m+1}^k)-\lambda(T_{m-1}^k)}{4}; \quad (15)$$

$$d_m=\frac{\gamma c(T_m^k)}{\Delta\tau}\Delta\rho^2T_m^k. \quad (16)$$

Making use of the last equation of (10), we eliminate the unknown T_{M-1} from Eqs. (8) and (9), which after some simple transformations become

$$T_M=a_1f_1(T_M,\ T_f)+a_2,$$
$$T_f=b_1f_2(T_M,\ T_f)+b_2, \quad (17)$$

where

$$a_1=\frac{\Delta\rho}{\lambda(1-x_M)};\quad a_2=\frac{y_M}{1-x_M}; \quad (18)$$

$$b_1=\frac{\Delta\tau}{1+\alpha_5\Delta\tau};\quad b_2=\frac{T_f^k+\Delta\tau(\alpha_5\Phi+\alpha_6u^{k+1})}{1+\alpha_5\Delta\tau};$$

$$f_1(T_M,\ T_f)=\alpha_1\left[T_f^4-\left(\frac{1+x_M}{2}T_M+\frac{y_M}{2}\right)^4\right]+$$

$$+\alpha_2\left[T_f-\left(\frac{1+x_M}{2}T_M+\frac{y_M}{2}\right)\right];$$

$$f_2(T_M,\ T_f)=\alpha_3\left[T_f^4-\left(\frac{1+x_M}{2}T_M+\frac{y_M}{2}\right)^4\right]+$$

$$+\alpha_4\left[T_f-\left(\frac{1+x_M}{2}T_M+\frac{y_M}{2}\right)\right]. \quad (19)$$

We can now solve our system as follows. Computing the coefficients x_1, y_1, a_m, b_m, c_m, d_m (m = 1, 2, ..., M − 1) from formulas (11), (13)–(16), we obtain the sweep coefficients $x_2 \ldots x_M$; $y_2 \ldots y_M$ in accordance with (12), and, solving system (17), determine T_M^{k+1}, T_f^{k+1}.

We then use formula (10) to find the temperature distributions at (k + 1) instants, computing T_{M-1}, ..., T_1, T_0 successively.

Computations according to this scheme are always correct, since the error present in y_m and T_m is multiplied in the course of computations by a coefficient smaller than unity in absolute value [3].

The most vulnerable point of the described scheme is the solution of system (17), which generally can have four pairs of distinct roots. We tested three methods of solving system (17): the Newton method, the iteration method, and Seidel's method [6]. The second and third of these methods afforded good convergence to the required solution in all of the examples computed. The iterative solution of system (17) by Seidel's method, which we shall consider because of its more rapid convergence, was based on the formulas

$$T_M^{i+1}=a_1f_1(T_M^i,\ T_f^{i+1})+a_2,$$
$$T_f^{i+1}=b_1f_2(T_M^i,\ T_f^i)+b_2, \quad (20)$$

where i is the number of the iteration. The computations were terminated upon simultaneous fulfillment of the conditions $|T_f^{i+1}-T_f^i|\le\varepsilon$, $|T_M^{i+1}-T_M^i|\le\varepsilon$; all of the computations with $\varepsilon=10^{-3}$ required from three to ten steps of the iteration process before attainment of the required degree of accuracy.

As an example we present the results obtained for the heating of a foam fireclay cylinder in a cylindrical electric muffle furnace. The thermophysical parameters were as follows: c(T) = 1.163(0.2080 + + 0.0001T) W/kg · degree; λ(T) = 1.163(0.6100 + + 0.0006T) W/m · degree; γ = 1800 kg/m³; α_1 = = 0.314 · 10^{-7}; α_3 = 0.630 · 10^{-8}; $\alpha_2=\alpha_4=0$; α_5 = = 6.70; α_6 = 8120; R = 0.022 m; τ_0 = 0.25 hr; T is the temperature, °K.

Figure 2 shows the function u(τ) and also the theoretically determined temperatures at the axis and surfaces of the cylinder and the furnace temperature. The computations were carried out for $\Delta\tau=10^{-3}$ hr; $\Delta\rho=10^{-3}$ m; 2 · 10^{-3} m.

The errors were estimated by the Runge method based on a comparison of the theoretical results obtained for differing numbers of layers in the net domain [5]. The error ε_{M_1} admitted in computing with an M_1-layer net is given by the formula

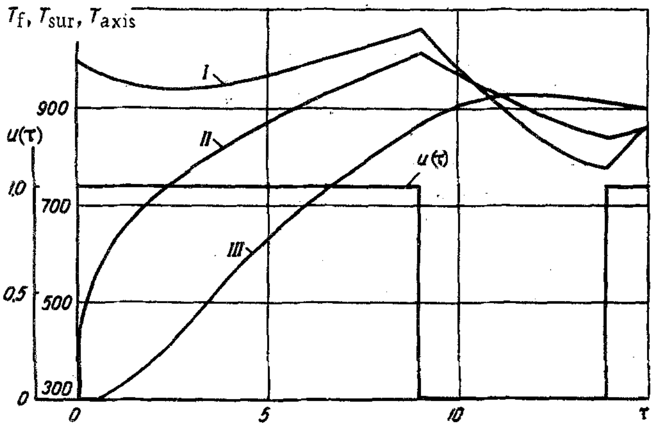

Fig. 2. Theoretical results for a foam fireclay cylinder heated in a cylindrical electric furnace ($u(\tau)$ is the electric furnace power as a function of time expressed in arbitrary units). I) T_f; II) T_{sur}; III) T_{axis}.

$$\varepsilon_{M_1} = T_{M_1} - T_{M_2} \Big/ \left(\frac{M_1}{M_2}\right)^2 - 1, \qquad (21)$$

where T_{M_1} and T_{M_2} are the temperatures at some node of the net for M_1 and M_2 layers, respectively.

Our computations for 10 and 20 layers indicated that the error for M = 10 was not more than 0.5° over the entire computed time interval.

Computations by our method are stable for practically all $\Delta\tau$. This makes it suitable for the construction of fast computer algorithms for solving the heat conduction problem. Analogous procedures can be used to compute the symmetric heating of a slab, sphere, and orthogonal parallelepiped.

NOTATION

$T(\rho, \tau)$ is the temperature distribution in the cylinder; T_{sur} and T_f are the body surface and furnace temperatures, respectively; $\lambda(T)$, $c(T)$ and γ are the thermal conductivity, specific heat, and density of the material; τ is the time; τ_0 is the reciprocal heating time; R is the cylinder radius; α_1, α_2, α_3, α_4, α_5, and α_6 are certain constants characterizing heat transfer in the "furnace-heated body" system; $\Delta\rho$ is the distance between the nodes of the computation net; m = 0, 1, ... M are the node numbers of the computation net; k = 1, 2, ..., N are the numbers of the computation time intervals; T_f^0 is the initial furnace temperature; Φ is the temperature of the medium around the furnace; T_{axis} is the temperature at the cylinder axis.

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12 October 1966

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